

## Electron trapping by a current vortex

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1998 J. Phys. A: Math. Gen. 31 L305

(<http://iopscience.iop.org/0305-4470/31/17/001>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.121

The article was downloaded on 02/06/2010 at 06:36

Please note that [terms and conditions apply](#).

## LETTER TO THE EDITOR

**Electron trapping by a current vortex**F Bentosela<sup>†‡¶</sup>, P Exner<sup>§+</sup> and V A Zagrebnov<sup>†‡\*</sup><sup>†</sup> Centre de Physique Théorique, CNRS, F-13288 Marseille Luminy, France<sup>‡</sup> Université de la Méditerranée (Aix-Marseille II), F-13288 Luminy, France<sup>§</sup> Nuclear Physics Institute, Academy of Sciences, CZ-25068 Řež near Prague, Czech Republic<sup>||</sup> Doppler Institute, Czech Technical University, Břehová 7, CZ-11519 Prague, Czech Republic

Received 10 February 1998, in final form 16 March 1998

**Abstract.** We investigate an electron in the plane interacting with the magnetic field due to an electric current forming a localized rotationally symmetric vortex. We show that independently of the vortex profile an electron with spin antiparallel to the magnetic field can be trapped if the vortex current is strong enough. In addition, the electron scattering on the vortex exhibits resonances for any spin orientation. On the other hand, in distinction to models with a localized flux tube this situation exhibits no bound states for weak vortices.

Interaction of charged particles with a localized magnetic field has been a subject of interest for a long time, both from the theoretical and experimental point of view—see, for example, [GBG, Št] and references therein. Such a field can have different sources, for instance, it may be induced by an electric current having one or more vortices. In the last decade a lot of attention has been paid to vortex bound states in superconductors whose dynamics is governed by the Bogoliubov–de Gennes equation—cf [HRD, SHDS, GS] and a bibliography given in [HIM].

Another, much simpler, example involves a Pauli electron interacting with a flux tube modelling a vortex magnetic field—it is appealing, in particular, since it has been observed that vortices often appear in the probability current associated with mesoscopic transport—see [EŠSF] and the literature there. The system of a tube and an electron has been investigated in a recent paper by Cavalcanti *et al* [CFC] who, however, seem to be unaware of another recent studies of the problem—cf [Mo] and references therein. In these papers, the field is assumed to be constant within a circle and zero otherwise—the conclusion then is that in one spin state the electron can always be trapped by the vortex, independently of the magnetic flux value, as long as the effective gyromagnetic factor  $g^* > 2$ . This covers the physically important case of a free electron with  $g^* = 2.0023$ .

However, this claim depends substantially on the magnetic field ansatz used. To illustrate this point we analyse in this letter the situation where the *vortex current distribution* represents the input. On one hand we are able to generalize the result of [Mo, CFC] by showing that vortices can trap electrons independently of their profile, and what is physically equally important, that they cause resonances in the electron–vortex scattering which become

¶ E-mail address: bentosela@cpt.univ-mrs.fr

+ E-mail address: exner@ujf.cas.cz

\* E-mail address: zagrebnov@cpt.univ-mrs.fr

sharper as the vortex strength grows. On the other hand, the trapping needs, in this more realistic setting, a certain minimum vortex strength (measured by the total circulating current or, say, the dipole moment of the field) to occur. To make things simpler we suppose that the vortex is *centrally symmetric*. We are convinced that the symmetry is not vital for the conclusions, but its absence makes the treatment technically much more complicated and we postpone the discussion of the general case to another paper.

The dynamics of a nonrelativistic electron in the plane exposed to a perpendicular but nonhomogeneous magnetic field is given by the respective Pauli equation, i.e. by the Hamiltonian

$$H = \frac{1}{2m^*} \left( -i\hbar\nabla + \frac{e}{c}\mathbf{A}(\mathbf{x}) \right)^2 + \frac{1}{2}g\mu_B\sigma_3 B(\mathbf{x}) \quad (1)$$

where  $m^*$  is the effective electron mass,  $g$  the gyromagnetic factor,  $\mu_B = e\hbar/2m_e c$  the Bohr magneton, and the sign choice corresponds to the negative charge  $-e$  of the electron. In the rational units,  $2m^* = \hbar = c = 1$  this becomes

$$H = (-i\nabla + e\mathbf{A}(\mathbf{x}))^2 + \frac{1}{2}g^*e\sigma_3 B(\mathbf{x}) \quad (2)$$

with the effective gyromagnetic factor  $g^* = g\frac{m^*}{m_e}$ . Since  $H$  is matrix-diagonal, each spin state can be treated separately. Moreover, we shall consider the situation when the magnetic field is rotationally symmetric. This allows us to perform the partial-wave decomposition and to replace  $H$  by the family of operators

$$H_\ell^{(\pm)} = -\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr} + V_\ell^{(\pm)}(r) \quad V_\ell^{(\pm)}(r) := \left( eA(r) + \frac{\ell}{r} \right)^2 \pm \frac{1}{2}g^*eB(r) \quad (3)$$

on  $L^2(\mathbb{R}^+, r dr)$ . To determine their spectral properties, we have to specify the involved functions, the angular component  $A(r)$  of the vector potential and the related magnetic field,  $B(r) = A'(r) + r^{-1}A(r)$ .

We have said that they correspond to a circulating electric current in the plane, which is supposed to be anticlockwise and to have the angular component only,  $\mathbf{J}(\mathbf{x}) = \lambda\delta(z)J(r)e_\varphi$ . Here  $r, \varphi$  are the polar coordinates in the plane; the total current is  $\lambda\int_0^\infty J(r)dr$ . The positive parameter  $\lambda$  can, of course, be absorbed into  $J$ ; we introduce it as a tool to control the vortex 'strength'. The current density is supposed to obey the following modest requirements:

- (i)  $J$  is  $C^2$  smooth and non-negative  $J(r) \geq 0$
- (ii) at the origin  $J(r) = ar^2 + \mathcal{O}(r^3)$
- (iii) at large distances  $J(r) = \mathcal{O}(r^{-3-\epsilon})$  for some  $\epsilon > 0$ .

The vector potential in the plane  $z = 0$  is then also anticlockwise; its magnitude

$$A(r) = \lambda \int_0^\infty dr' r' J(r') \int_0^{2\pi} \frac{\cos \varphi' d\varphi'}{(r^2 + r'^2 - 2rr' \cos \varphi')^{1/2}} \quad (4)$$

is obtained by summing the contribution from all the circular current lines [Ja],

$$A(r) = \lambda \int_0^\infty J(r') \frac{4r}{r+r'} \frac{(2-\rho^2)K(\rho^2) - 2E(\rho^2)}{\rho^2} dr' \quad (5)$$

where  $\rho^2 := \frac{4rr'}{(r+r')^2}$  and  $K, E$  are the full elliptic integrals of the first and the second kind, respectively. Using [AS, 17.3.29, 30] (pay attention to a misprint there), one can also cast (5) into the form

$$A(r) = 4\lambda \int_0^\infty J(r') \frac{r'}{r_<} \left[ K\left(\frac{r_<^2}{r_>^2}\right) - E\left(\frac{r_<^2}{r_>^2}\right) \right] dr' \quad (6)$$

where we have used the usual shorthands,  $r_< := \min(r, r')$  and  $r_> := \max(r, r')$ ; the same result follows directly from (4) and [GR, 3.674.3]. In view of (i) the integral is finite for every  $r$ , because  $E(\zeta)$  is regular at  $\zeta = 1$  and  $K(\zeta)$  has a logarithmic singularity there.

Let us denote the Pauli Hamiltonian (2) with the vector potential (6) by  $H(\lambda)$ ; the symbol  $H_\ell^{(\pm)}(\lambda)$  will be used for its spin and orbital momentum components. Our main result is then the following:

*Theorem.* Under the stated assumptions,  $\sigma(H(\lambda)) = [0, \infty)$  for  $|\lambda|$  small enough. On the other hand,  $H_0^{(-)}(\lambda)$  has a negative eigenvalue for a sufficiently large  $\lambda$ .

*Proof.* The first claim has to be checked for any  $H_\ell^{(\pm)}(\lambda)$ . By (i), the potentials  $V_\ell^{(\pm)}$  are  $C^1$  smooth; they decay at infinity as

$$V_\ell^{(\pm)}(r) = \frac{\ell^2}{r^2} + \lambda em \frac{2\ell \mp g^*}{r^3} + \mathcal{O}(r^{-3-\epsilon}) \quad (7)$$

where  $m := \pi \int_0^\infty J(r')r'^2 dr'$  is the dipole moment of the current for  $\lambda = 1$ —cf [Ja]. Consequently,  $\sigma_{ess}(H_\ell^{(\pm)}(\lambda)) = [0, \infty)$  following [RS, section XIII.4]. We rewrite the potentials into the form

$$V_\ell^{(\pm)}(r) = \left( \lambda e A_1(r) + \frac{\ell}{r} \right)^2 \pm \frac{\lambda}{2} g^* e B_1(r) \quad (8)$$

where the indexed magnetic field refers to the value  $\lambda = 1$ . Since  $H_\ell^{(\pm)}(\lambda)$  is just the  $s$ -wave part of the two-dimensional Schrödinger operator with the centrally symmetric potential (8), it is sufficient to check that the latter has no negative eigenvalues. If  $\ell = 0$  the potential decay allows us to apply theorem 3.4 of [Si] by which a negative eigenvalue exists for small positive  $\lambda$  if and only if  $\int_0^\infty V_\ell^{(\pm)}(r)r dr \leq 0$ . However, the flux through the circle of radius  $r$  is  $2\pi \int_0^r B(r')r' dr' = 2\pi r A(r)$ , so the second term in (8) does not contribute and the integral is determined by the first one which is positive for any nonzero  $\lambda$ . If  $\ell \neq 0$  the decay is too slow, but this difficulty is easily overcome. We replace the first term, for example, by  $(\lambda e A_1(r) + \frac{\ell}{r})^2 \Theta(r_0 - r)$  with a positive  $r_0$  and obtain the absence of a negative eigenvalue for a small  $\lambda$ ; the same is, by the minimax principle, true for the original operator.

For the existence claim the behaviour of the potential around the origin is vital. We shall write the vector potential in the form

$$A(r) = \lambda \mu r + \alpha_0(r) \quad \mu := \int_0^\infty J(r') \frac{dr'}{r'}. \quad (9)$$

The behaviour of  $\alpha_0$  follows from (6) and the following estimate on the difference of the elliptic integrals,

$$\frac{\pi \zeta}{4} \leq K(\zeta) - E(\zeta) \leq \frac{\pi}{2} \left( \frac{\zeta}{8} - \frac{3}{8} \ln(1 - \zeta) + b \zeta^2 \right) \quad (10)$$

for a sufficiently large  $b > 0$ , which is a straightforward consequence of [AS, 17.3.11, 12] and [GR, 8.123.2]. On the lower side we thus obtain

$$\alpha_0(r) \geq \frac{\pi}{r^2} \int_0^r J(r')r'^2 dr' - \pi r \int_0^r J(r') \frac{dr'}{r'} \geq -\frac{3}{4} \pi a r^2 + \mathcal{O}(r^3).$$

The upper bound is similar. In view of (i)–(iii),  $J(r) \leq \tilde{a}r$  for a suitable  $\tilde{a}$ ; in the logarithmic term we employ the Taylor expansion which gives

$$\frac{3\pi\tilde{a}}{4r} \int_r^\infty \sum_{j=2}^\infty \frac{r^{2j}}{(r')^{2j-2}} dr' = \frac{3\pi\tilde{a}}{4} r^2 \sum_{j=2}^\infty \frac{1}{j(2j-3)}.$$

Together we find  $\alpha_0(r) = \mathcal{O}(r^2)$ . This further implies

$$B(r) = 2\lambda\mu + \beta_0(r) \quad \beta_0(r) := \alpha_0'(r) + \frac{1}{r}\alpha_0(r) \in \mathcal{O}(r). \quad (11)$$

The operator  $H_0^{(-)}(\lambda)$  can therefore be written as

$$H_0^{(-)}(\lambda) = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + (\lambda e(\mu r + \alpha_0(r)))^2 - \frac{1}{2}\lambda e g^*(2\mu + \beta_0(r)). \quad (12)$$

Using the rescaled variable  $u := r\sqrt{\lambda}$  we find it to be unitarily equivalent to the operator

$$\lambda A_\lambda \quad \text{with } A_\lambda = A_0 + W_\lambda \quad (13)$$

on  $L^2(\mathbb{R}^+, u du)$ , where

$$A_0 := -\frac{d^2}{du^2} - \frac{1}{u} \frac{d}{du} - g^* e\mu + e^2 \mu^2 u^2 \quad (14)$$

and

$$W_\lambda(u) := 2\sqrt{\lambda} e^2 \mu u \alpha_0\left(\frac{u}{\sqrt{\lambda}}\right) + \lambda e^2 \alpha_0^2\left(\frac{u}{\sqrt{\lambda}}\right) - \frac{1}{2} g^* e \beta_0\left(\frac{u}{\sqrt{\lambda}}\right). \quad (15)$$

We have clearly  $\sigma_{ess}(A_\lambda) = \sigma_{ess}(\lambda A_\lambda) = [0, \infty)$  for any  $\lambda > 0$ , while  $A_0$  as the  $s$ -wave part of the two-dimensional harmonic oscillator has a purely discrete spectrum. Despite the fact that the perturbation is large (the maximum of  $W_\lambda$  grows linearly with  $\lambda$ ), one may attempt to employ the *asymptotic* perturbation theory.

Unfortunately, theorems VIII.3.11 and VIII.13 of [Ka] cannot be applied directly, because the family  $\{W_\lambda\}$  is not monotonous. Instead we use the fact that  $W_\lambda$  tends to zero pointwise as  $\lambda \rightarrow \infty$ , since the above estimates yield

$$|W_\lambda(u)| \leq (2e^2 \mu c_\alpha u^3 + \frac{1}{2} g^* e c_\beta u) \lambda^{-1/2} + e^2 c_\alpha^2 u^4 \lambda^{-1} \quad (16)$$

for some positive  $c_\alpha, c_\beta$ . The family  $\{A_\lambda\}$  can be estimated from below because the potentials have in view of (15) a uniform lower bound. Hence one can choose  $\xi_0 < 0$  which belongs to  $\rho(A_\lambda)$  for all  $\lambda$ , and the resolvents form a bounded family of positive operators,  $(A_\lambda - \xi)^{-1} \leq (\xi_0 - \xi)^{-1}$ , for any  $\xi < \xi_0$ . Next we use a trick based on the resolvent identity: for a vector  $f = (A_0 - \xi)g$  with a fixed  $g \in C_0^\infty(\mathbb{R}^+)$  we have

$$\|(A_\lambda - \xi)^{-1} f - (A_0 - \xi)^{-1} f\| = \|(A_\lambda - \xi)^{-1} W_\lambda g\| \leq (\xi_0 - \xi)^{-1} \|W_\lambda g\| \rightarrow 0 \quad (17)$$

as  $\lambda \rightarrow \infty$  in view of (16) and the compact support of  $g$ . However, the family of such  $f$  is dense in  $L^2(\mathbb{R}^+, u du)$ , so  $A_\lambda \rightarrow A_0$  in the strong resolvent sense.

This allows us to employ theorem VIII.1.14 of [Ka] by which to any  $v_n \in \sigma_p(A_0)$  there is a family of  $v_n(\lambda) \in \sigma(A_\lambda)$  such that  $v_n(\lambda) \rightarrow v_n$  as  $\lambda \rightarrow \infty$ . It is therefore decisive that the unperturbed eigenvalue is stable in the sense of [Ka], which means negative in our case. The spectrum of  $A_0$  is given explicitly by

$$v_n = e\mu(4n + 2 - g^*) \quad n = 0, 1, \dots \quad (18)$$

so the condition is satisfied for  $n = 0$  if  $g^* > 2$  (as in [CFC], the next eigenvalue comes in to play for  $g^* > 6$  which is not physically appealing).  $\square$

We finish the letter by remarks on extensions of the result and related topics.

(a) The difference in the *weak-coupling behaviour* comparing with [Mo, CFC] is not surprising. In their case  $\int_0^\infty V_0^{(-)}(r)r dr$  is dominated for small  $\lambda > 0$  by the negative term due to the well. In reality, however, the magnetic field flux lines are closed in  $\mathbb{R}^3$ , so the well is compensated for by a repulsive tail, small but extended, which prevents the trapping.

(b) The asymptotic perturbation theory yields also the ground-state behaviour as  $\lambda \rightarrow \infty$ . By [Ka, theorem VIII.2.6] the leading-order correction to  $\nu_0$  is

$$(\psi_0, W_\lambda \psi_0) = \frac{3e}{8}(g^* + 2)\sqrt{\frac{\pi}{e\mu}}\alpha_0''(0)\lambda^{-1/2} + \mathcal{O}(\lambda^{-1})$$

where  $\psi_0$  is the ground-state eigenfunction of the two-dimensional harmonic oscillator. However,  $\alpha_0''(0) = 0$ , so the ground state of the original operator  $H_0^{(-)}(\lambda)$  behaves as  $-\lambda e\mu(g^* - 2) + \mathcal{O}(1)$ .

(c) Large  $\lambda$  give rise to an *orbital series* of bound states: the above argument also works for  $H_\ell^{(-)}(\lambda)$  with  $\ell = -1, -2, \dots$ . The potential in (14) is replaced at that by  $e^2\mu^2u^2 + \ell^2r^{-2} + e\mu(2\ell - g^*)$ , and one looks for negative eigenvalues among  $\nu_{n,\ell} = e\mu(4n + 2(|\ell| + \ell) + 2 - g^*)$ . The critical  $\lambda$  at which the eigenvalue emerges from the continuum is naturally  $\ell$ -dependent.

(d) Positive eigenvalues of  $A_0$  are unstable in the sense that they disappear in the continuum once the perturbation (15) is turned on. Following [Ka, section VIII.5], however, they give rise to spectral concentration as  $\lambda \rightarrow \infty$  which is manifested by *resonances* in electron scattering on the vortex. Knowing the shape of the potential barrier, one can compute their widths which vanish exponentially fast with  $\lambda$ . For a fixed  $\ell$ , the number of resonances grows asymptotically linearly with  $\lambda$ , because the eigenvalues of the operator (14) are equally spaced and the top of the potential barrier in  $A_\lambda$  is asymptotically linear in  $\lambda$ . For the same reason, resonances exist at large  $\lambda$  for *both spin signs*. Note also that the existence of resonances is not restricted by the value of the gyromagnetic factor, and therefore they may be observed in semiconductor systems where  $|g^*|$  is typically less than one, in some cases even of the order of  $10^{-2}$ .

(e) The above-mentioned resonance scattering is a *purely quantum effect*. A classical electron can, of course, be trapped in the current vortex if it is placed inside the potential barrier (3) with the energy less than its top (since there is no spin in this case,  $g^* = 0$ , the well bottom is at zero). On the other hand, an electron of the same energy outside the barrier gets scattered without the possibility of entering temporarily the interior of the vortex.

(f) In addition to the smooth current distributions discussed above, the case  $J(r) = \delta(r - R)$  is of a practical interest. As an illustration, imagine two adjacent thin films, one supporting a free-electron gas while the other is equipped with a mesoscopic ring in which a persistent current circulates. The effective potential is now given explicitly by (2) and (6); in distinction to the smooth case it has a singularity of the type  $(r - R)^{-1}$  and the operators  $H_\ell^{(\pm)}(\lambda)$  need a regularization. Since the  $\delta$  function above is an idealization of a sharply localized distribution, it is reasonable to choose for this purpose from the family of all admissible procedures a scheme based on the principal value of the singular potential [NZ, Ku].

(g) The Pauli operator with a localized magnetic field exhibits the properties noted by Aharonov and Casher [AC, Th]. They are based on a factorization property of  $H$  valid for  $g^* = 2$ ; hence they do not conflict with the existence of a bound state with a preferred spin orientation established above. The existence of a critical

current needed to bind the electron can be also understood in terms of the first AC theorem. In contrast to [CFC] the total flux is zero so there are no zero-energy solutions for  $g^* = 2$  which could change into true bound states under an arbitrarily weak perturbation.

(h) Note finally another difference. The model with a flux tube can have a natural ‘squeezing limit’ in terms of Aharonov–Bohm Hamiltonians with a pointlike magnetic flux [Mo, AT, DŠ] because the vector potentials coincide outside the flux tube and the attractive part tends to a  $\delta$ -well (with the exception of a single one, the resonances mentioned in (d) are lost at that as is usual in such situations—cf [AGHH]). On the other hand, the present case is more complicated being essentially three-dimensional as far as the magnetic field is concerned.

A useful comment by R Cavalcanti is appreciated. PE is grateful for the hospitality extended to him at CPT where this work was done. The research has been partially supported by GAAS under the contract A1048801.

## References

- [AS] Abramowitz M S and Stegun I A (ed) 1965 *Handbook of Mathematical Functions* (New York: Dover)
- [AT] Adami R and Teta A On the Aharonov–Bohm Hamiltonian *Preprint mp\_arc97-37*
- [AC] Aharonov Y and Casher A 1979 Ground state of a spin-1/2 charged particle in a two-dimensional magnetic field *Phys. Rev. A* **19** 2641–2
- [AGHH] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (Berlin: Springer)
- [CFC] Cavalcanti R M, Fraga E S and de Carvalho C A A 1997 Electron localization by a magnetic vortex *Phys. Rev. B* **56** 9243–6
- [DŠ] Dabrowski L and Štoviček P 1998 Aharonov–Bohm effect with the  $\delta$  type interaction *J. Math. Phys.* **39** 47–62
- [EŠSF] Exner P, Šeba P, Sadreev A F, Středa P and Feher P 1998 Strength of topologically induced magnetic moments in a quantum device *Phys. Rev. Lett.* **80** 1710–13
- [GBG] Geim A, Bending S J and Grigorieva I V 1992 Asymmetric scattering and diffraction of two-dimensional electron at quantized tubes of magnetic flux *Phys. Rev. Lett.* **69** 2252–5
- [GR] Gradshteyn I S and Ryzhik I M 1980 *Tables of Integral, Series, and Products* (New York: Academic)
- [GS] Gygi F and Schluter M 1990 Angular band structure of a vortex line in a type-II superconductor *Phys. Rev. Lett.* **65** 1820–3
- [HIM] Hayashi N, Ichioka M and Machida K 1997 Effects of gas anisotropy upon the electronic structure around a superconducting vortex *Phys. Rev. B* **56** 9052–63
- [HRD] Hess H F *et al* 1989 Scanning-tunneling-microscope observation of the Abrikosov flux lattice and the density of states near and inside a fluxoid *Phys. Rev. Lett.* **62** 214–16
- [Ja] Jackson J D 1962 *Classical Electrodynamics* (New York: Wiley)
- [Ka] Kato T 1966 *Perturbation Theory for Linear Operators* (Heidelberg: Springer)
- [Ku] Kurasov P 1997 Response to ‘Comment . . .’ by Fischer, Leschke and Müller *J. Phys. A: Math. Gen.* **30** 5583–9
- [Mo] Moroz A 1996 Single-particle density of states, bound states, phase-shift flip, and a resonance in the presence of an Aharonov–Bohm potential *Phys. Rev. A* **53** 669–94
- [NZ] Neidhardt H and Zagrebnov V A 1996 Towards the right Hamiltonian for singular perturbations via regularization and extension theory *Rev. Math. Phys.* **8** 715–40
- [RS] Reed M and Simon B 1978 *Methods of Modern Mathematical Physics, IV. Analysis of Operators* (New York: Academic)
- [SHDS] Shore J D, Ming Huang, Dorsey A T and Sethna J P 1989 Density of states in a vortex core and zero-bias tunneling peak *Phys. Rev. Lett.* **62** 3089–92

- [Si] Simon B 1976 The bound state of weakly coupled Schrödinger operators in one and two dimensions  
*Ann. Phys.* **97** 279–88
- [Št] Šťovíček P 1994 Scattering on a finite chain of vertices *Duke Math. J.* **76** 303–32
- [Th] Thaller B 1992 *The Dirac Equation* (Berlin: Springer)